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MORE ON CUMULATIVE SEARCH EVASION GAMES

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This report generalizes the form of the payoff function so that all track crossings must involve contact. A more computationally efficient form of the one-dimensional game is also given.

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MORE ON CUMULATIVE SEARCH EVASION GAMES

1. INTRODUCTION

Eagle and Washburn (1990) introduced Cumulative Search Evasion Games (CSEGs) as two-person zero sum games where the cumulative payoff over T time periods is $\sum_{t=1}^{T} A(x_t, y_t, t)$, x_t and y_t being the locations of searcher and evader, respectively, at time t. A path for the searcher is a sequence $x_1, \ldots, x^T$ where $x_1 \in S_0$ and $x_{t+1} \in S(x_t,t)$ for $t \geq 1$, the sets S_0 and $S(\bullet,\bullet)$ being given, and similarly for the evader except $y_1 \in E_0$ and $y_t \in E(y_t, t)$. All of these sets are nonempty subsets of Cs a given finite set of "cells." A mixed strategy for the searcher is a probability distribution over paths. Let $p(x,t)$ be the corresponding marginal distribution, the probability that the searcher occupies cell x at time t, and let $q(y,t)$ be defined similarly for the evader. Then the expected payoff is $\sum_{t=1}^{T} \sum_x \sum_y A(x, y, t) p(x,t) q(y,t)$. This observation, together with the observation that the optimization problem for one player when the marginal distribution of the other is given is a shortest or longest-path problem, formed the basis of two solution methods for solving CSEGs: Fictitious Play and Linear Programming (LP). Only the LP method will be discussed here.

One might hope to formulate an LP for the searcher in which the only variables needed to describe the searcher's mixed strategy are $p(x,t)$, since those suffice to express the expected payoff. However, Eagle and Washburn found it necessary to introduce the joint probabilities

$u(i, j, t) =$ probability that the searcher occupies cell i at time $t{-}1$, and cell j at time t,

together with network constraints to the effect that probabilities "flowing" into and out of a cell must balance. The necessity to include these joint probabilities is disappointing, since in large problems there are many more u-variables than p-variables. One of the goals of this paper is to show that the u-variables can be avoided in certain one-dimensional CSEGs. This is the subject of the next section. Using only the p-variables makes it possible to solve larger CSEGs than would otherwise be possible.

The other goal of this paper is to show that the payoff at time t in a CSEG can be generalized to $A(x_{t-1}, x_t, y_{t-1}, y_t, t)$ if the u-variables are retained. The required theorems and LP formulation, together with an example illustrating the value of the generalization, is the subject of Section 3.

2. THE ONE-DIMENSIONAL CSEG

In this section the positions of both parties must at all times be in the set of cells $C = \{1, ..., N\}$, $N \geq 1$, with transitions from i to j at t being permissible if $i \in C$, $j \in C$, and $|i - j| \leq 1$. These rules define $E(\bullet,\bullet)$ and $S(\bullet,\bullet)$. The payoff function $A(i, j, t)$ is unrestricted.

Suppose for the moment that the searcher's marginal probabilities $p(i, t)$ were known to the evader, in which case any evader path that visits cell j at time t must pay a penalty ("penalty" because the evader is the minimizer) of $\sum_{i \in C} p(i,t)A(i,j,t)$. Let $g(j, t)$ be the minimum possible cumulative payoff from time t onwards, given that the evader occupies cell j at time t. Then, taking $g(\bullet, T + 1) \equiv 0$ for convenience, $g(\bullet, \bullet)$ must satisfy the recursion

$$g(j,t) = \sum_{i \in C} p(i,t)A(i,j,t) + \min_{k \in E(j,t)} g(k,t+1); j \in C, 1 \le t \le T \tag{1}$$

Since the evader must be in E_0 at time 1, the minimum possible payoff is $\min_{j \in E_0} g(j,1)$, which the pursuer wants to maximize. This leads to the following Linear Program:

$$\text{maximize } g_0$$

subject to $\quad g_0 - g(y,1) \le 0; \quad j \in E_0,$

$$g(j,t) - \sum_{i \in C} p(i,t)A(i,j,t) - g(k,t+1) \le 0; j \in C, 1 \le t \le T, k \in E(j,t),$$

and some feasibility constraints on $p(\bullet, \bullet)$.

Eagle and Washburn employed the u– variables in expressing the feasibility constraints on $p(\bullet, \bullet)$. The object here is to find a way of expressing those constraints without defining any new variables. First we prove

Theorem 1. In the one-dimensional CSEG, $p(\bullet, \bullet)$ is feasible if the following feasibility constraints hold:

$$\sum_{i \in S_0} p(i,1) = 1$$

(left) $\quad \sum_{i=1}^{k} p(i,t+1) - \sum_{i=1}^{k+1} p(i,t) \le 0 \quad ;1 \le k < N; 1 \le t < T$

(right) $\quad \sum_{i=k+1}^{N} p(i,t+1) - \sum_{i=k}^{N} p(i,t) \le 0 \quad ;1 \le k < N; 1 \le t < T$

$$\sum_{i=1}^{N} p(i,t) = 1 \quad ;1 < t \le T$$

$$p(i,t) \ge 0 \quad ;1 \le i \le N; 1 \le t \le T$$

Proof: Assume that the feasibility constraints hold, and consider the proposition P_T that there exists a feasible stochastic searcher motion process for which the marginal distributions are $p(\bullet,t)$; $1 \le t \le T$. P_1 is clearly true, since the feasibility constraints in that case require only that the searcher begin in S_0. If it can be shown that P_T implies P_{T+1}, the theorem will be established by induction. Toward this end, let cells 1, ..., N at time T be "sources" with probability $p_i \equiv p(i, T)$ each, and let the same cells at time $T + 1$ be "sinks" with probability $q_i \equiv p(i, T + 1)$ each. To establish P_{T+1}, it is sufficient to show that there exist N^2 joint occupancy probabilities u_{ij} such that $\sum_{i=1}^{N} u_{ij} = p_i$, $\sum_{j}^{N} u_{ij} = q_j$, and $u_{ij} = 0$ unless $j \in E(i, t)$, the latter constraint reflecting the requirement that transitions beyond neighboring cells are not allowed. In other words, it must be possible to "ship" a unit of probability from sources to sinks, with u_{ij} being the amount shipped from source i to sink j. The "left biased" method (LB) below is one constructive method for accomplishing this. LB proceeds through the sources in increasing order, shipping probability to the lowest numbered sink that is not yet satisfied until the source being considered is exhausted, then proceeding to the next source until all N sources have been considered. If LB makes $u_{ij} > 0$ for some i and some $j < i - 1$ (alternatively $j > i + 1$), we say that a left (alternatively right) difficulty occurs at node i. To complete the proof it is required to show that no difficulties of either type can occur as long as the feasibility constraints hold.

Suppose that no difficulties occur in cells 1, ..., $k - 1$, but that a left difficulty occurs in cell k (necessarily $k \ge 3$, since left difficulties are not possible in cells 1 and 2). Since all of the probability in sources 1, ..., k–1 can be shipped to sinks 1, ,..., $k - 2$ without satisfying one of those sinks (otherwise the left difficulty could not occur in cell k), necessarily

$$\sum_{i=1}^{k-2} q_i - \sum_{i=1}^{k-1} p_i > 0.$$

But this inequality is in the opposite sense of one of the left constraints, so a left difficulty cannot occur in cell k. Suppose instead that there is a right difficulty. A right difficulty occurs for the first time in cell k only if there is more probability in sources 1, ..., k than is required to satisfy sinks 1, ..., $k + 1$, so

$$\sum_{i=1}^{k+1} q_i < \sum_{i=1}^{k} p_i.$$

Since (p_i) and (q_i) are both constrained to be probability distributions, it follows that

$$\sum_{i=k+2}^{N} q_i - \sum_{i=k+1}^{N} p_i > 0.$$

But this contradicts one of the right constraints, so right difficulties cannot occur either.

Since neither right nor left difficulties can occur, LB will discover a feasible set of joint probabilities u_{ij}. This completes the proof. □

Obviously there is a symmetrically defined "right-biased method" that will discover a possibly different set of feasible joint probabilities. In fact there are many such methods and many feasible sets of joint probabilities. Formulating the searcher's linear program without reference to these joint probabilities has the advantage of eliminating many alternate optima, in addition to the computational savings achieved by eliminating variables. The revised formulation, with dual variables shown in braces, is program LP:

$$\text{maximize}\, g_0$$

$$\text{subject to} \quad g_0 - g(j,1) \le 0 \qquad ; j \in E_0 \quad \{q(j,1)\}$$

$$g(j,t) - \sum_{i=1}^{N} p(i,t)A(i,j,t) - g(k,t+1) \le 0 \qquad ; j \in C, 1 \le t < T, k \in E(j,t) \quad \{v(j,k,t+1)\}$$

$$g(j,T) - \sum_{i=1}^{N} p(i,T)A(i,j,T) \le 0 \qquad ; j \in C \quad \{q(j,T)\}$$

$$\sum_{i \in S_0} p(i,1) = 1 \qquad ; \{h_1\}$$

$$\sum_{i=1}^{k} p(i,t+1) - \sum_{i=1}^{k+1} p(i,t) \le 0 \qquad ; 1 \le k < N, 1 \le t < T \quad \{l(k,t)\}$$

$$\sum_{i=k+1}^{N} p(i,t+1) - \sum_{i=k}^{N} p(i,t) \le 0 \qquad ; 1 \le k < N, 1 \le t < T \quad \{r(k,t)$$

$$\sum_{i=1}^{N} p(i,t) = 1 \qquad ; 1 < t \le T \quad \{h_t\}$$

It has been established so far that the value, v, of the CSEG is at least g_0. The possibility still remains that $v > g_0$. To establish $v = g_0$, the dual of LP will be shown to be a Linear Program whose objective function is an upper bound on the game value. Consideration of the dual will also provide interpretations of the dual variables in LP; the notation used above anticipates that $q(j,1)$ can be interpreted as the probability described earlier, for example, but that fact has yet to be established formally.

The dual of LP involves the sums $\sum_{k=i}^{N} l(k,t) \equiv L(i,t)$ and $\sum_{k=1}^{i} \mathrm{r(k,t)} \equiv R(i,t)$. For compactness we will write $L(\bullet,\bullet)$ and $R(\bullet,\bullet)$ below, even though the sums are actually meant, and we will also use the convention that $L(0,t) \equiv L(1,t)$ and $R(N+1,t) \equiv R(N,t)$. Note that, since $l(\bullet,\bullet)$ and $r(\bullet,\bullet)$ are nonnegative, $L(\bullet,t)$ and $R(\bullet,t)$ are nonincreasing and nondecreasing cell functions, respectively, for $1 \leq t < T$. Finally, the set $E^*(i,t)$ consists of those cells from which the evader at time $t-1$ can transition to cell i at time t. The dual of LP is DLP:

$$\text{minimize} \sum_{t=1}^{T} h_t$$

$$\text{subject to } h_1 - \sum_{j=1}^{N} A(i,j,1) \sum_{k \in E(j,1)} v(j,k,2) - L(i-1,1) - R(i+1,1) \geq 0 \qquad ; i \in S_0 \qquad \{p(i,1)\}$$

$$h_t - \sum_{j=1}^{N} A(i,j,t) \sum_{k \in E(j,t)} v(j,k,t+1) + L(i,t-1) + R(i,t-1) - L(i-1,t) - R(i+1,t) \geq 0 \qquad ; i \in C, 1 < t < T \qquad \{p(i,t)\}$$

$$-\sum_{j=1}^{N} A(i,j,T) q(j,T) + L(i,T-1) + R(i,T-1) \geq 0 \qquad ; i \in C \qquad \{p(i,T)\}$$

$$\sum_{j \in E(i,t)} v(i,j,t+1) - \sum_{k \in E^*(i,t)} v(k,i,t) = 0 \qquad ; i \in C, 1 < t < T \qquad \{g(i,t)\}$$

$$q(k,T) - \sum_{j \in E^*(k,T)} v(j,k,T) = 0 \qquad ; k \in C \qquad \{g(k,T)\}$$

$$\sum_{k \in E(j,1)} v(j,k,2) - q(j,1) = 0 \qquad ; j \in C \qquad \{g(j,1)\}$$

$$\sum_{i \in E_0} q(i,1) = 1 \qquad ; \qquad \{g_0\}$$

$$v(i,j,t) \geq 0; \quad l(i,t) \geq 0; \quad r(i,t) \geq 0; \quad q(i,1) \geq 0; \quad q(j,T) \geq 0$$

The last four sets of constraints in DLP have the effect of requiring that $q(\bullet,\bullet)$ be a feasible marginal distribution for the evader, with $v(\bullet,\bullet,\bullet)$ being the joint occupancy probabilities. The first three sets of constraints can be simplified somewhat by defining $y(i,t) \equiv \sum_{j=1}^{N} A(i,j,t) \sum_{k \in E(j,t)} v(j,k,t+1)$, so that $y(i,t)$ is the average payoff to the searcher at time t if he occupies cell i at that time, and also $L(\bullet,T) = R(\bullet,T) = 0$. In that case the first three sets of constraints can be summarized as

$$h_1 - y(i,1) - L(i-1,1) - R(i+1,1) \geq 0 \qquad ; i \in S_0 \qquad (2)$$

$$h_t - y(i,t) + L(i,t+1) + R(i,t-1) - L(i-1,t) - R(i+1,t) \geq 0 \qquad ; i \in C, 1 < t \leq T \quad (3)$$

The question now is, "Do (2) and (3) guarantee that the accumulated payoff is at most $\sum_{t=1}^{T} h_t$ for any feasible searcher path?" Theorem 2 answers this question in the affirmative.

Theorem 2: Suppose that (2) and (3) hold, with $L(\bullet,t)$ and $R(\bullet,t)$ being nonincreasing and nondecreasing functions, respectively, on $\{1, \ldots, N\}$, and $L(\bullet,T) = R(\bullet,T) = 0$. Let $x_1, \ldots, x_T$ be any sequence of integers such that $x_1 \in S_0$, $1 \leq x_t \leq N$ for $1 \leq t \leq T$, and $|x_t - x_{t-1}| \leq 1$ for $t > 1$. Then $\sum_{t=1}^{N} y(x_t,t) \leq \sum_{t=1}^{N} h_t$.

Proof: Substitute x_t for i in the tth inequality of (2)–(3), and sum all T inequalities. The result is

$$\sum_{t=1}^{T} h_t - \sum_{t=1}^{T} y(x_t,t) + \sum_{t=2}^{T} [L(x_t,t-1) - L(x_{t-1}-1,t-1)] + [R(x_t,t-1) - R(x_{t-1}+1,t-1)] \geq 0 \tag{4}$$

Since $L(\bullet,t-1)$ is nonincreasing and since $x_t \geq x_t-1$, $L(x_t,t-1) - L(x_{t-1}-1,t-1) \leq 0$ for $t = 2, \ldots, T$. Similarly $R(x_t, t-1) - R(x_{t-1}+1,t-1) \leq 0$. Therefore the third sum in (4) is nonpositive, and the theorem follows directly. □

Theorem 2 implies that the optimized g_0 from LP is the value of the CSEG, as well as providing probabilistic interpretations for the dual variables $q(i, 1)$, $v(j,k,t+1)$, and $q(i, T)$. Thus the value of the game and both optimal strategies can be obtained from LP.

Bothwell (1990) reports on some experiments in using LP as above (as well as other methods) to solve a one-dimensional CSEG where $A(i, j, t)$ indicates whether $i = j$, so that the payoff is "total number of coincidences," with $S_0 = \{1\}$ and $E_0 = \{N\}$. He discovered that the new formulation permitted solutions in about one fourth of the time of the Eagle-Washburn method, and was thus able to solve games up to $N = 30$. His Figures 1-6 describe the solution for $N = 20$ and $T = 31$. The searcher's strategy $p(\bullet,\bullet)$ is shown digitally in Figure 1 and graphically in Figure 2. Figure 3 is a blowup for $t \geq 21$, showing that $p(\bullet,31)$ is finally uniform, that $p(1,t)$ goes through a maximum, and that $p(20,t)$ goes through a minimum. The latter two features were unanticipated, but seem to be regular features of the solution for large N. Basically the searcher "rushes" from cell 1 to cell 20, except that he has a small probability of reversing his direction after time 10. The cumulative effect of all the small probabilities is to make $p(\bullet,t)$ uniform for $t = 31$.

Figures 4-6 show the evader's marginal probabilities $q(\bullet,\bullet)$. Basically the evader stays in cell 20, except that there is at all times (even $t = 1$) a small probability of making a break for the other side; one is reminded of Auger's

		CELLS 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6	0	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	0	0	0	0	0	0	1000	0	0	0	0	0	0	0	0	0	0
	11	0	0	0	0	0	0	0	0	0	500	500	0	0	0	0	0	0	0	0	0
	12	0	0	0	0	0	0	0	0	22	10	491	477	0	0	0	0	0	0	0	0
	13	0	0	0	0	0	0	0	10	22	0	13	477	477	0	0	0	0	0	0	0
	14	0	0	0	0	0	0	10	9	13	13	19	13	470	452	0	0	0	0	0	0
T	15	0	0	0	0	0	10	9	13	13	19	13	19	22	441	441	0	0	0	0	0
I	16	0	0	0	0	10	9	13	13	15	17	19	22	42	10	415	415	0	0	0	0
M	17	0	0	0	10	5	15	15	15	17	19	22	24	28	33	33	382	382	0	0	0
E	18	0	0	10	5	15	15	15	17	19	22	24	28	33	33	41	41	342	341	0	0
	19	0	10	5	15	15	15	17	19	22	24	28	33	33	41	41	52	52	290	290	0
	20	0	15	15	15	15	17	19	22	24	28	33	33	41	41	52	52	84	55	220	220
	21	15	15	15	15	17	19	22	24	28	33	33	41	41	52	52	66	73	147	147	147
	22	19	19	19	19	19	22	24	28	33	33	41	41	52	52	66	73	110	110	110	110
	23	24	24	24	24	24	24	28	33	33	41	41	52	52	66	73	88	88	88	88	88
	24	28	28	28	28	28	28	33	33	41	41	52	52	66	73	73	73	73	73	73	73
	25	34	34	34	34	34	34	34	41	41	52	52	64	64	64	64	64	64	64	64	64
	26	39	39	39	39	39	39	41	41	52	52	58	58	58	58	58	58	58	58	58	58
	27	45	45	45	45	45	45	45	52	52	58	58	52	52	52	52	52	52	52	52	52
	28	53	53	53	53	53	53	53	53	58	58	46	46	46	46	46	46	46	46	46	46
	29	56	56	56	56	56	56	56	56	56	45	45	45	45	45	45	45	45	45	45	45
	30	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
	31	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50

Figure 1. Searcher Marginal Probabilities (×1000) for 20-Cell CSEG

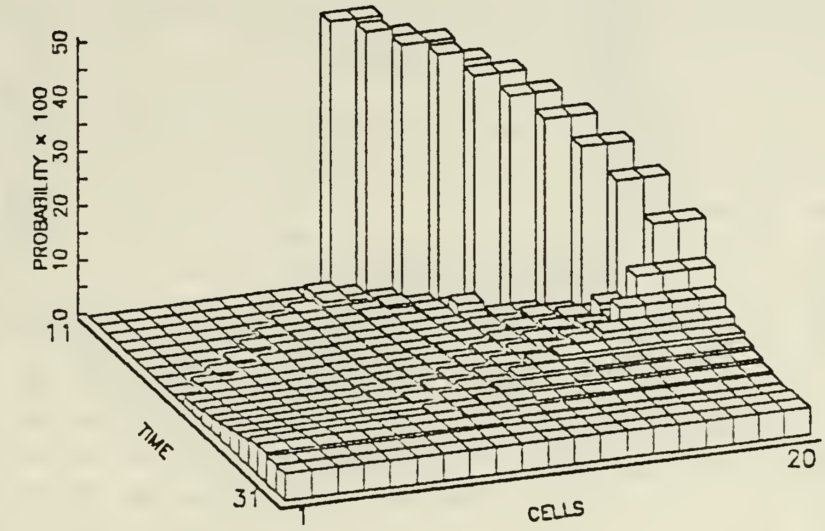

Figure 2. Searcher Strategy

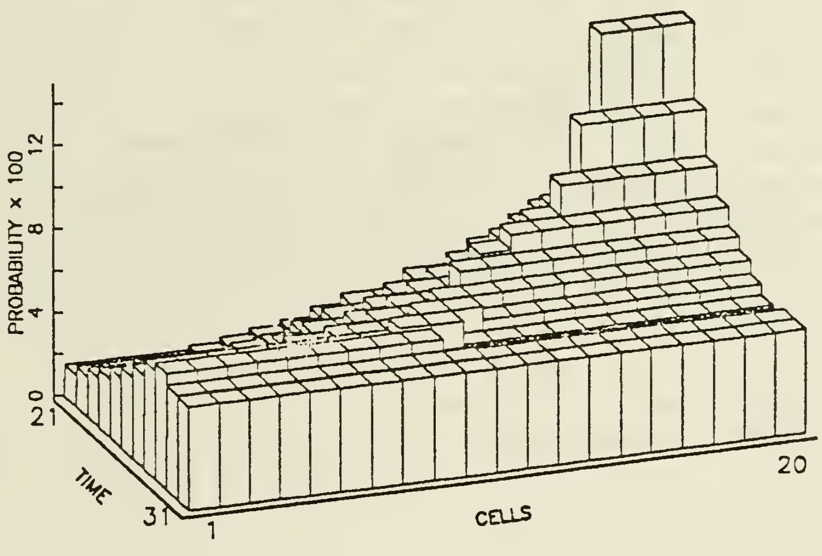

Figure 3. Searcher Strategy—Final Time Periods

		CELLS																			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1000
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	964
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	36	928
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	36	36	892
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	36	36	36	856
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	36	36	36	36	37	819
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	36	36	36	36	37	37	782
	8	0	0	0	0	0	0	0	0	0	0	0	0	36	36	36	36	37	37	39	743
	9	0	0	0	0	0	0	0	0	0	0	0	36	36	36	36	37	37	39	39	704
	10	0	0	0	0	0	0	0	0	0	0	36	36	36	36	37	37	39	39	41	663
	11	0	0	0	0	0	0	0	0	0	36	36	36	36	37	37	39	39	41	41	622
	12	0	0	0	0	0	0	0	0	36	36	36	36	37	37	39	39	41	41	44	578
	13	0	0	0	0	0	0	0	36	36	36	36	37	37	39	39	41	41	44	44	535
	14	0	0	0	0	0	0	36	36	36	36	37	37	39	39	41	41	44	44	49	485
T	15	0	0	0	0	0	36	36	36	36	37	37	39	39	41	41	44	44	49	49	436
I	16	0	0	0	0	36	36	36	36	37	37	39	39	41	41	44	44	49	49	54	382
M	17	0	0	0	36	36	36	36	37	37	39	39	41	41	44	44	49	49	54	54	329
E	18	0	0	36	36	36	36	37	37	39	39	41	41	44	44	49	49	54	54	121	208
	19	0	36	36	36	36	37	37	39	39	41	41	44	44	49	49	54	54	60	60	208
	20	36	36	36	36	37	37	39	39	41	41	44	44	49	49	54	54	60	60	104	104
	21	48	48	48	37	37	39	39	41	41	44	44	49	49	54	54	60	60	69	69	69
	22	54	54	54	54	39	39	41	41	44	44	49	49	54	54	60	60	52	52	52	52
	23	59	59	59	59	59	41	41	44	44	49	49	54	54	60	60	42	42	42	42	42
	24	63	63	63	63	63	63	44	44	49	49	54	54	60	60	35	35	35	35	35	35
	25	66	66	66	66	66	66	66	49	49	54	54	37	37	37	37	37	37	37	37	37
	26	78	78	78	78	78	78	49	49	39	39	36	36	36	36	36	36	36	36	36	36
	27	68	68	68	68	68	68	68	45	45	39	39	40	40	40	40	40	40	40	40	40
	28	59	59	59	59	59	59	59	59	45	45	44	44	44	44	44	44	44	44	44	44
	29	53	53	53	53	53	53	53	53	53	48	48	48	48	48	48	48	48	48	48	48
	30	47	47	47	47	47	47	47	47	47	47	53	53	53	53	53	53	53	53	53	53
	31	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50

Figure 4. Evader Marginal Probabilities (×1000) for 20-Cell CSEG

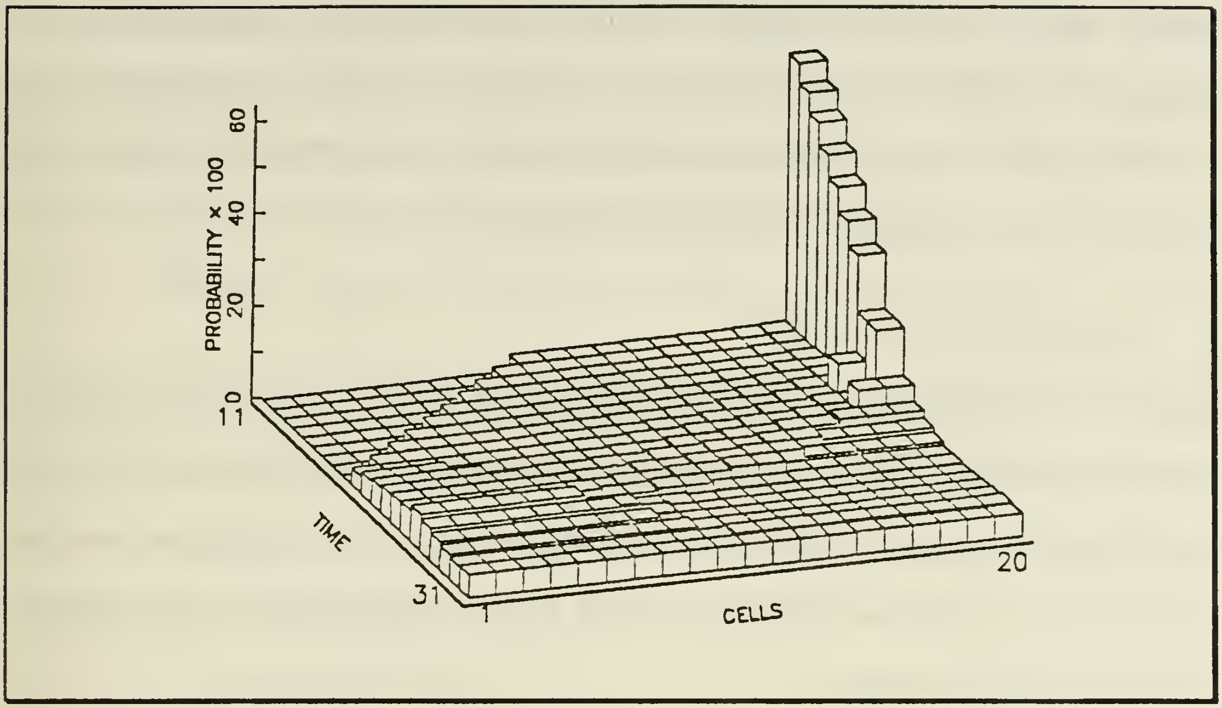

Figure 5. Evader Strategy

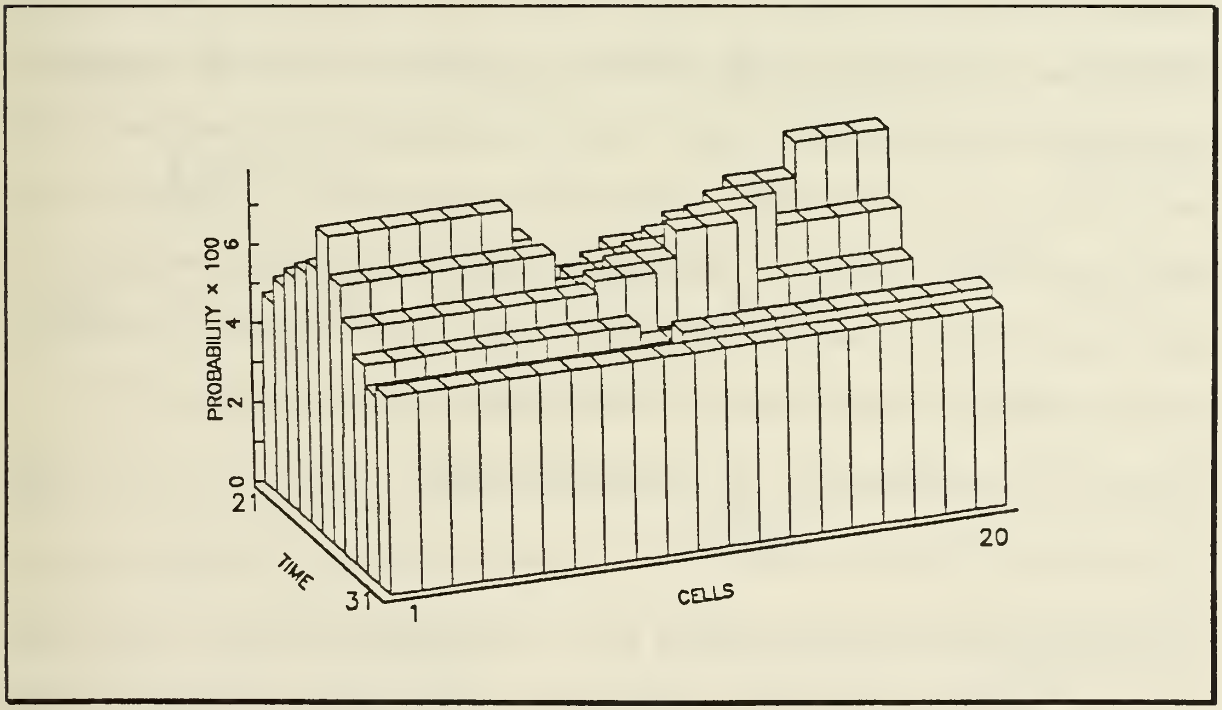

Figure 6. Evader Strategy—Final Time Periods

(1991) "Wait-and-run" strategies. By time 31 the evader's position, like the searcher's, is uniform over all 20 cells. It follows (see Eagle and Washburn) that the game where $T > 31$ starts the same way as when $T = 31$, but that it is optimal for each player to remain stationary for $31 \le t \le T$.

3. GENERALIZED PAYOFF

In this section it will be shown that the payoff in a CSEG can be generalized to $\sum_{t=1}^{T} A(x_{t-1}, x_t, y_{t-1}, y_t, t)$, with x_0 and y_0 specified. Solution of such games will require retention of the joint occupancy probabilities, so the contribution of this section is toward modeling flexibility, rather than computational efficiency.

Let $S_0 = \{x_0\}$, $E_0 = \{y_0\}$, and let $S(\bullet,\bullet)$ and $E(\bullet,\bullet)$ be as defined in Section 1 except that $S(x_0,0)$ and $E(y_0,0)$ are now (rather than S_0 and E_0) the sets of cells feasible for searcher and evader at Time 1. S_0 and E_0 are now the (singleton) sets of cells feasible at time 0. For $t \ge 1$ let S_t be the set of cells feasible for the searcher at time t. Formally, $S_t = \{j$: there exists i in S_{t-1} such that j is in $S(i, t-1)\}$. Define E_t similarly. Also, for $t \ge 1$ and $j \in S_t$, let $S^*(j, t)$ be the set of cells from which j is feasible, formally $S^*(j, t) = \{i$: $j \in S(i, t-1)\}$, and define $E^*(\bullet,\bullet)$ similarly. Finally, let $u(\bullet,\bullet,\bullet)$ be as defined as in Section 1, so that

$$f(m,n,t) \equiv \sum_{\substack{i \in S_{t-1} \\ j \in S(i,t-1)}} A(i,j,m,n,t)u(i,j,t); \quad 1 \le t \le T \tag{5}$$

is the penalty at time t to the evader if he occupies cell m at time $t-1$ and cell n at time t, and $\sum_{t=1}^{T} f(y_{t-1}, y_t, t)$ is the total expected penalty, conditioned on the evader's track.

Consider first the evader's problem of minimizing the total penalty when $u(\bullet,\bullet,\bullet)$ is known. A dynamic programming recursion is still feasible. Let $h(m,t)$ be the minimum total penalty over periods $t, \ldots, T$ if the evader occupies cell m at time $t-1$. Then $h(m,t)$ satisfies the recursion

$$h(m,t) = \min_{n \in E(m,t-1)} \{f(m,n,t) + h(n,t+1)\}; \quad 1 \le t \le T, m \in E_{t-1} \qquad (6)$$

with $h(\bullet,T+1) \equiv 0$. The minimized total penalty over all T periods is then $h(y_0,1)$, which quantity the searcher wants to maximize. Since (6) can be written as linear constraints, maximizing $h(y_0,1)$ is a linear program. The program, with dual variables named in braces as usual, is LP1:

$$\text{maximize}\, h(y_0,1)$$

subject to

$$-f(m,n,t) - h(n,t+1) + h(m,t) \le 0 \quad ; 1 \le t \le T, m \in E_{t-1}, n \in E(m,t-1) \quad \{v(m,n,t)\}$$

$$\sum_{j \in S_1} u(x_0,j,1) = 1 \quad ; \quad \{g(x_0,1)\}$$

$$-\sum_{j \in S^*(i,t)} u(j,i,t) + \sum_{k \in S(i,t)} u(i,k,t+1) = 0 \quad ; 1 \le t < T, i \in S_t \quad \{g(i,t+1)\}$$

$$u(i,j,t) \ge 0 \quad ; 1 \le t \le T, i \in S_{t-1}, j \in S(i,t-1).$$

In LP1 $f(m, n, t)$ has been written for compactness, even though the expression on the right-hand side of (5) is meant, and it should be understood that the term $h(n, t+1)$ is missing when $t = T$. The second and third sets of constraints are the feasibility constraints of Eagle and Washburn; as long as $u(\bullet,\bullet,\bullet)$ satisfies those constraints, there exists a feasible mixed strategy for the searcher with $u(\bullet,\bullet,\bullet)$ as the joint occupancy probabilities. Thus any feasible solution to LP1 corresponds to a lower bound $h\,(y_0,1)$ on the value of the CSEG, and consequently the same thing can be said of the maximized value.

LP1 and its dual DLP1 possess a pleasing symmetry that was absent in Section 2. DLP1 is (the $g(j,t+1)$ term is missing when $t = T$)

$$\text{minimize}\, g(x_0,1)$$

subject to

$$-\sum_{\substack{m\in E_{t-1}\\ n\in E(m,t-1)}} A(i,j,m,n,t)v(m,n,t) - g(j,t+1) + g(i,t) \geq 0$$

$$;1 \leq t \leq T, i \in S_{t-1}, j \in S(i,t-1) \quad \{u(i,j,t)\}$$

$$\sum_{n\in E_1} v(y_0,n,1) = 1 \qquad \{h(y_0,1)\}$$

$$-\sum_{n\in E^*(m,t)} v(n,m,t) + \sum_{k\in E(m,t)} v(m,k,t+1) = 0 \qquad ;1 \leq t < T, m \in E_t \qquad \{h(m,t+1)\}$$

$$v(m,n,t) \geq 0 \qquad ;1 \leq t \leq T, m \in E_{t-1}, n \in E(m,t-1).$$

Any function $v(\bullet,\bullet,\bullet)$ that meets the second and third sets of constraints of DLP1 can be interpreted as the joint occupancy probabilities of a feasible mixed strategy for the evader. That being the case, the first set of constraints assures that a searcher in cell i at time $t-1$ cannot obtain a payoff larger than $g(i,t)$ over periods $t, \ldots, T$. In particular, $g(x_0,1)$ is an upper bound on the cumulative payoff over all T periods. But the optimized values of $g(x_0,1)$ and $h(y_0,1)$ must be equal because LP1 and DLP1 are duals, so either number is the value of the CSEG. Furthermore the evader's optimal occupancy probabilities can be obtained as the dual variables associated with the first set of constraints in LP1; it is actually not necessary to solve DLP1.

Example: The revised one-dimensional CSEG

In the standard one-dimensional CSEG described earlier, it is possible that the two tracks $x_1, \ldots, x_T$ and $y_1, \ldots, y_T$ may cross each other without ever being exactly coincident, in which case the searcher's score will be 0 because the objective function simply counts coincidences. To guard against this possibility, the searcher's leading edge as he moves from 1 to N is spread into two approximately equal parts, thus making a barrier so wide that the evader cannot "jump over it" (see Figure 2). This annoying artifact can be eliminated by redefining the payoff so that the searcher scores a point whenever the two tracks cross, even if they are never exactly coincident. Specifically, for $1 \leq i,j \leq N$ let

$$A(i,j,m,n,t) = \begin{cases} 1 & \text{if } j = n \\ 1 & \text{if } i = n \text{ and } j = m \\ \text{otherwise} & 0 \end{cases} \qquad (7)$$

Figure 7 shows a GAMS program (Brooke, Kendrick, and Meeraus, 1988) to solve a 10 cell CSEG with payoff (7) where the initial moves of searcher and evader are from cells 4 to 5 and 7 to 6, respectively. Figure 8 shows the associated output. The value of the CSEG is 1.2269 (scaled to 122.69 in Figure 8), to be compared with .8541 in the "standard" game where $A(i,j,m,n,t)$

```
OPTIONS SOLPRINT=OFF,ITERLIM=5000,LIMROW=0,LIMCOL=0
SET
    I /C1*C10/
    T /T1*T10/
    E(I,I,T) HOLDS FEASIBLE TRANSITIONS FOR EVADER
    S(I,T) HOLDS FEASIBLE CELLS FOR SEARCHER
    SS(I,I,T) HOLDS FEASIBLE TRANSITIONS FOR SEARCHER
ALIAS (I,J,K);
*** SEARCHER STARTS BY MOVING FROM HALF-1 TO HALF
*** EVADER STARTS BY MOVING FROM HALF+2 TO HALF+1
PARAMETER
    HALF;
HALF=FLOOR(.5*CARD(I));
E(I,J,T)=YES$(ABS(ORD(I)-ORD(J)) LE 1 AND ORD(I)+ORD(T) GT HALF+2
    AND ORD(J)+ORD(T) GT HALF+1);
E(I,J,T)$(ORD(I) GE HALF+ORD(T))=NO;
E(I,J,"T1")$(ORD(I) EQ HALF+2 AND ORD(J) EQ HALF+1)=YES;
S(I,T)=YES$(HALF+ORD(T) GT ORD(I));
S(I,T)$(HALF GE ORD(I)+ORD(T))=NO;
SS(I,J,T+1)$S(I,T)=YES$(ABS(ORD(I)-ORD(J)) LE 1);
SS(I,J,"T1")=YES$(ORD(J) EQ HALF AND ORD(I) EQ HALF-1);
VARIABLES
    H(I,T)
    U(I,J,T)
    Z;
POSITIVE VARIABLE U;
U.FX(I,J,"T1")=100$(ORD(I) EQ HALF-1 AND ORD(J) EQ HALF);
EQUATIONS
    NDET
    BAL(I,T)
    OPT(I,J,T);
NDET..  Z=E=SUM(I$(ORD(I) EQ HALF+2),H(I,"T1"));
BAL(I,T+1)$S(I,T).. SUM(J$SS(I,J,T+1),U(I,J,T+1))
              -SUM(K$SS(K,I,T),U(K,I,T))=E=0;
OPT(I,J,T)$(E(I,J,T))..H(I,T)-H(J,T+1)
     -SUM(K$SS(K,J,T),U(K,J,T))
     -U(J,I,T)$SS(J,I,T)$(ORD(J) NE ORD(I))=L=0;
MODEL LINESEARCH /ALL/;
SOLVE LINESEARCH USING LP MAXIMIZING Z;
OPTION DECIMALS=4;
DISPLAY H.L;
PARAMETER
    P(I,T)  MARGINALS
    Q(J,T)  EVADER MARGINALS
    G(J,T) ROUTE TOTALS SEEN BY PURSUER;
P(I,T)$S(I,T)=SUM(K$SS(K,I,T),U.L(K,I,T));
Q(J,T)=100*SUM(I$E(I,J,T),OPT.M(I,J,T));
G(J,T)=100*BAL.M(J,T);
DISPLAY P,Q,G;
MODEL STATISTICS

BLOCKS OF EQUATIONS      3     SINGLE EQUATIONS    253
BLOCKS OF VARIABLES      3     SINGLE VARIABLES    254
NON ZERO ELEMENTS     .300
```

Figure 7. One-Dimensional Crossing Game

---- 43 VARIABLE H.L

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
C1							18.1478	16.4247	12.0387	7.2676
C2						18.6177	18.1478	16.4247	13.7618	7.4895
C3					19.0876	19.4009	19.0876	16.4247	13.7618	8.7688
C4				20.8890	21.9855	21.7506	19.0876	17.7562	13.7618	8.7688
C5			22.6904	23.2387	24.4135	21.7506	21.7506	18.7548	16.2583	8.7688
C6		122.6904	24.4918	25.7450	24.4135	25.7450	23.7478	23.7478	18.1307	7.4462
C7	122.6904		122.6904	31.0709	29.7394	28.7408	31.2373	27.4925	20.3429	7.4462
C8				122.6904	33.7338	38.7268	36.8544	37.9605	25.0639	17.6177
C9					122.6904	46.2163	55.5782	37.9605	25.0639	13.2132
C10						122.6904	73.1959	58.9366	35.2353	13.2132

---- 51 PARAMETER P MARGINALS

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
C1						0.4699	1.7231	4.3860	4.7711	7.2676
C2					0.4699	1.2531	2.6629	2.6629	6.2723	7.4895
C3				1.8014	1.2531	2.6629	2.6629	3.9944	4.9930	8.7688
C4			1.8014	1.2531	2.6629	2.6629	3.9944	4.9930	7.4895	8.7688
C5	100.0000	1.8014	1.2531	1.3315	2.6629	3.9944	4.9930	7.4895	9.3619	8.7688
C6		98.1986	5.3259	2.6629	3.9944	4.9930	7.4895	9.3619	12.8967	7.4462
C7			91.6196	3.9944	4.9930	7.4895	9.3619	17.6177	12.8967	7.4462
C3				88.9566	7.4895	9.3619	17.6177	12.8967	7.4462	17.6177
C9					76.4741	17.6177	35.2353	12.8967	11.8506	13.2132
C10						49.4946	14.2593	23.7013	22.0221	13.2132

---- 51 PARAMETER Q EVADER MARGINALS

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
C1						9.0836	16.3505	20.3073	20.3073	20.3073
C2					9.0836	7.2669	9.0836	13.7125	20.3073	20.3073
C3				9.0836	7.2669	9.0836	8.5858	9.6112	12.3057	20.3073
C4			9.0836	7.2669	9.0836	8.5858	9.6112	9.2893	9.8926	6.3029
C5		9.0836	7.2669	9.0836	8.5858	9.6112	9.2893	9.8926	10.7149	6.3029
C6	100.0000	7.2669	9.0836	8.5858	9.6112	9.2893	9.8926	10.7149	5.0423	6.3029
C7		83.6495	8.5858	9.6112	9.2893	9.8926	10.7149	10.0846	5.0423	5.0423
C8			65.9801	9.2893	9.8926	10.7149	13.8664	5.0423	6.3029	5.0423
C9				47.0797	10.7149	13.8664	6.3029	5.0423	5.0423	5.0423
C10					26.4722	12.6058	6.3029	6.3029	5.0423	5.0423

---- 51 PARAMETER G ROUTE TOTALS SEEN BY PURSUER

	T2	T3	T4	T5	T6	T7	T8	T9	T10
C1					93.6229	77.2725	60.9220	40.6147	20.3073
C2				122.6904	86.3561	77.2725	60.9220	40.6147	20.3073
C3			122.6904	95.4397	84.5394	70.0056	54.3272	40.6147	20.3073
C4		122.6904	104.5233	91.8062	79.0892	62.9131	50.2259	32.6130	20.3073
C5	122.6904	113.6068	99.0731	88.1728	71.4989	59.8370	41.9023	30.1999	6.3029
C6		106.3400	97.2564	80.0847	69.4482	51.1915	40.0924	17.0178	6.3029
C7			88.6706	79.0594	60.4808	49.9850	27.7328	11.3452	6.3029
C8				69.7701	59.8775	38.4477	21.4299	11.3452	5.0423
C9					49.1626	35.2962	16.3875	11.3452	5.0423
C10						22.6904	16.3875	10.0846	5.0423

Figure 8. One-Dimensional Crossing Game

simply indicates whether $j = n$. The prohibition of scoreless crossovers is evidently a significant change in the rules of the game. Note that the leading edge of the searcher's marginals (P) is now only 1 cell wide for $1 \le t \le 6$.

The revised game differs qualitatively in an interesting way from the standard game. Let $v_N(T)$ and $v_N'(T)$ be the values of the standard and revised games (so $v_{10}(10) = .8541$ and $v_{10}'(10) = 1.2269$). $v_N(T)$ is ultimately linear in T with slope $1/N$. For example $v_{10}(T) = 1.2269 + (T-10)/10$ for $T \ge 10$. The turnpike theorem of Eagle and Washburn makes this plausible; essentially either side can guarantee a slope of $1/N$ by remaining stationary in a randomly chosen cell. Stationarity has the same virtues in the revised game, but there is no evidence that $v_N'(T)$ is ultimately linear. For $T = (12, 14, 16, 18, 20)$, $v_{10}'(T)$ is (1.4486, 1.7109, 2.000, 2.1396, 2.3540). The differences fluctuate about .2, but are never exactly equal to .2. It is possible, of course, that $T = 20$ is simply not large enough to observe the onset of linearity.

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